WAVE SCATTERING CALCULATIONS FROM MULTIPLE INCLUSIONS USING A VOLUME INTEGRAL EQUATION

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INTRODUCTION

Elastic wave scattering problems have important applications in a variety of engineering fields including NDE. Scattering problems have been investigated by numerous authors with different solution schemes. For simple geometries of the scatterers (e.g., cylinders or spheres), the analysis of steady–state elastic wave scattering has been carried out using analytical techniques [1, 9]. For arbitrary geometries and multiple inclusions, numerical methods have been developed [2]. Special finite element methods, e.g., the infinite element method and Global–Local finite element method [3] have also been developed for this purpose. Recently, the boundary integral equation method has been used successfully to solve scattering problems [4–5]. In this paper, a volume integral equation (VIE) method is proposed as a new numerical solution scheme for the solution of general elastodynamic problems involving single or multiple inclusions sketched in Fig. 1. The relative advantages of the boundary and volume integral methods in solving multiple inclusion problems are discussed.

VOLUME AND BOUNDARY INTEGRAL EQUATION METHODS

The volume integral equation method is based on an integral representation of the scattered field originally derived by Mal and Knopoff [6] in the form

$$u_m(x) = u_m^0(x) + \int_{V_s} [\delta\rho\,\omega^2 G_i^m(x,\xi) u_i(\xi) - \delta C_{ijkl} G_{i,j}^m(x,\xi) u_{k,l}(\xi)] d\xi \tag{1}$$

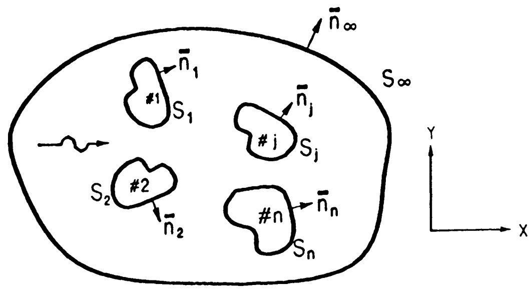


Fig. 1. Geometry of the multiple inclusion problem

In (1), **x** can be either inside or outside the inclusion, and $\delta\rho$ and δC_{ijkl} are the differences in the densities and the elastic constants between the inclusion and the matrix, respectively, $\delta\rho = \rho_1 - \rho_2$; $\delta C_{ijkl} = C_{ijkl}^{(1)} - C_{ijkl}^{(2)}$. The derivatives in the integrand are with respect to the integration variable ξ . The displacement $u_m^o(\mathbf{x})$ is the incident field from infinity and the integral represents the scattered field. $G_i^m(\mathbf{x}, \xi)$ is the Green's function for the unbounded host medium, i.e., the mth component of the displacement at **x** due to a unit time harmonic point force at ξ in the ith direction. In this paper, only the SH wave case will be considered for simplicity.

Boundary Integral Equation Method (BIE)

For SH waves, the integral equation on the outer surface of the inclusion can be expressed as

$$w(x) = w_o(x) - \int_S \mu_2 (G_2 \frac{\partial w}{\partial n} - w \frac{\partial G_2}{\partial n}) ds \tag{2}$$

while for the interior surface, the equation is

$$w(x) = \int_S \mu_1 (G_1 \frac{\partial w}{\partial n} - w \frac{\partial G_1}{\partial n}) ds \tag{3}$$

where μ_1 and μ_2 are the shear moduli, G_1 and G_2 are Green's functions for an unbounded inclusion and an unbounded matrix, respectively.

As the observation point approaches the interface, w(**x**) in (2), (3) is replaced by (1/2)w(**x**) for a smooth interface, and the derivatives ($\partial G_1/\partial n$), ($\partial G_2/\partial n$) must be interpreted in the Cauchy principal value sense. Applying the continuity condition at the interface, followed by discretization, a system of algebraic equations for the interface displacements and tractions are obtained. Once the displacements and tractions at the interface are known, displacements and stresses everywhere can be calculated from (2) and (3).

Volume Integral Equation Method (VIE)

For SH waves, the volume integral equation (1) becomes

$$w(x) = w_o(x) + \int_R \{\delta\rho\omega^2 G_2(x,\xi)w(\xi) - \delta\mu[G_2(x,\xi)_{,\xi_1}w(\xi)_{,\xi_1} + G_2(x,\xi)_{,\xi_2}w(\xi)_{,\xi_2}]\}d\xi_1 d\xi_2 \quad (4)$$

in which $\delta\mu = \mu_1 - \mu_2$. Since the unknowns are the displacements and strains inside the inclusion, it is convenient to discretize this region by using standard finite elements, resulting in a system of linear equations for the unknown nodal displacements inside the inclusion. Therefore, the integro–differential equation (4) can be solved numerically.

NUMERICAL RESULTS AND DISCUSSION

The Green's function can be expressed as

$$G(x,\xi) = \frac{iH_o^{(1)}(k|x-\xi|)}{4\mu} \quad (5)$$

where $k = \omega/\beta$ is the wave number and β is the shear wave speed. As $|\mathbf{x}-\xi| \to 0$, the asymptotic behaviour of the Green's function is given by

$$G = \frac{i}{4\mu}[\frac{2}{\pi}i\ \ln(k|x-\xi|)] + O(1) \quad (6)$$

and

$$\frac{\partial G}{\partial \xi_1}, \frac{\partial G}{\partial \xi_2}, \frac{\partial G}{\partial n} \approx -\frac{2i}{\pi k|x-\xi|} + O(1) \quad (7)$$

Thus, special care must be taken for treating *ln* r and 1/r type singularities in the integrals [7–8]. For both methods, the direct integration schemes were used instead of the indirect method [4].

Single inclusion

In order to check the accuracy of the numerical methods the single cylindrical inclusion is considered first. For comparison, the analytical solution [9] was also obtained. Fig. 2 shows a typical modeling for the volume integral equation method. When ka = 8, the wavelength in the host medium is shorter than the radius of the inclusion. In VIE, it is necessary to discretize inside the inclusion only. Thus, there are no finite elements outside the inclusion. Standard 8–node quadrilateral and 6–node triangular finite elements were used. In BIE, quadratic elements were used. Fig. 3 shows the comparison between the anlaytical solution and the numerical solutions using BIE and VIE. It can be seen that there is excellent agreement between the anlaytical solution and the numerical solutions.

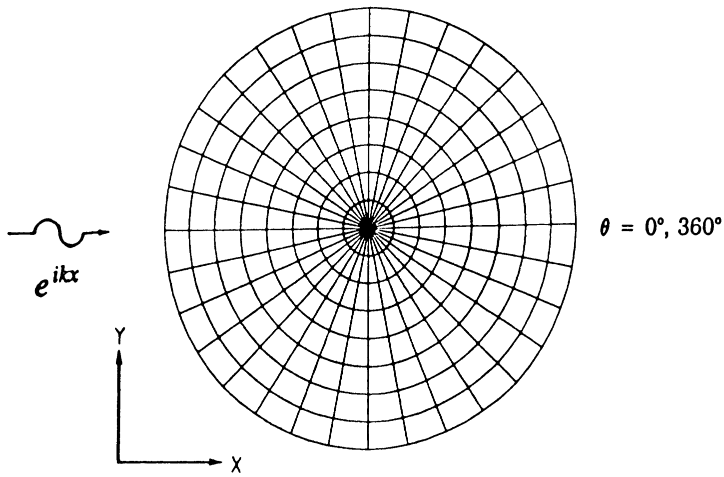


Fig. 2. A typical modeling in the volume integral equation. The radius of inclusion (a) is 2 cm, normalized wave number (ka) is 4 and 8. (k is the wavenumber in the host medium.)

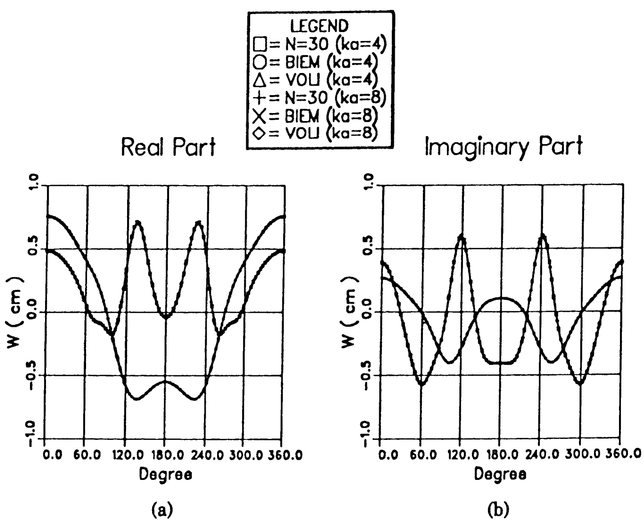


Fig. 3. Real part (a) and Imaginary part (b) of displacements around the interface (r = 2) for modeling in Fig. 2.

As an example of a noncircular shape, an elliptic inclusion is considered next. Fig. 4 shows the results obtained by means of the volume integral equation method and the boundary integral equation method; the agreement can be seen to be excellent. For a given frequency ω, a measure of the fraction of the incident power scattered in a particular direction is the differential cross section. Using the asymptotic behaviour of $H_0^{(1)}(k|\mathbf{x}-\xi|)$ and $H_1^{(1)}(k|\mathbf{x}-\xi|)$, as $|\mathbf{x}-\xi| \to \infty$, the far-field differential scattering cross-section can be calculated by using the volume integral equation. Fig. 5 shows different scattering cross-sections for three different shapes of the inclusions; the results are physically reasonable.

Multiple inclusions

Boundary Integral Equation Method

The unknowns are w, $\mu(\partial w/\partial n)$ for the host medium, and w_j , $\mu_j(\partial w_j/\partial n)$ at each interface. If N_j is the number of nodes on S_j, the total number of nodes at the interfaces is, $N = N_1 + N_2 + ... + N_n$. In order to solve for displacements and tractions at each interface, the following systems of equations can be considered. For the host medium,

$$[A]\{w\} + [B]\{\mu\frac{\partial w}{\partial n}\} = \{w_o\} \tag{8}$$

at the interfaces

$$[A_j]\{w_j\} + [B_j]\{\mu_j\frac{\partial w_j}{\partial n}\} = \{0\} \tag{9}$$

After enforcing the continuity conditions at each interface, a system of 2N equations is obtained.

Volume Integral Equation Method

Basically, there is no change in the formulation. In the host medium, $\delta\rho$ and $\delta\mu$ vanish. Therefore, it is necessary to discretize the inclusions only. For each observation point, it is required to integrate over the whole domain of each inclusion. By doing this, the interactions between all the inclusions can be modeled exactly. Furthermore, since the continuity condition at each interface is automatically satisfied in the formulation, it is very easy and convenient to use the volume integral equation method in multiple inclusions. So, in multiple inclusions, it was decided to use the volume integral equation method instead of the boundary integral equation method. The volume integral equation method can be applied to arbitrary packing sequence and shapes of the inclusions. However, in order to compare results in multiple inclusions with the available analytical solutions, simple packing sequence (hexagonal and square) and circular shape of the inclusions were considered. Fig. 6 shows the calculated average strains in the central fiber by analytical [9] and the volume integral equation methods for different number of inclusions. The percentage differeces in the two sets of results are less than 1 % in all cases. The code based on the VIE was modified to use one quarter of the medium by taking advantage of symmetry. It should be noted that in the volume integral equation

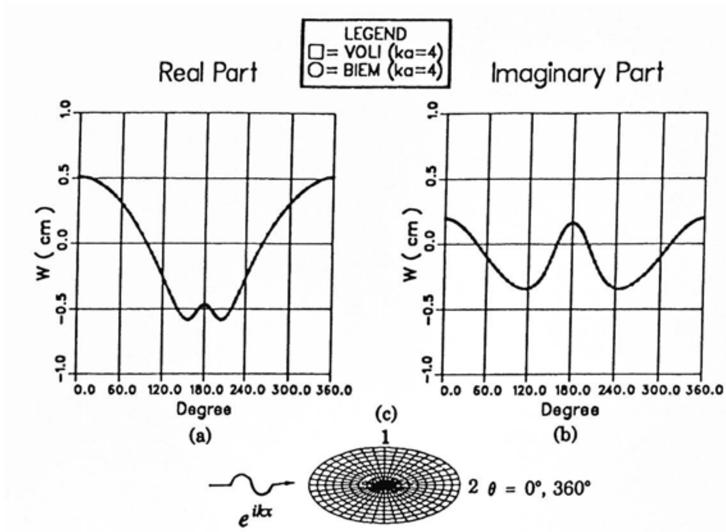


Fig. 4. Real part (a) and Imaginary part (b) of displacments around the interface for modeling in (c). Normalized wave number (ka) is 4. (k is the wavenumber in the host medium and a is the radius in the major axis.)

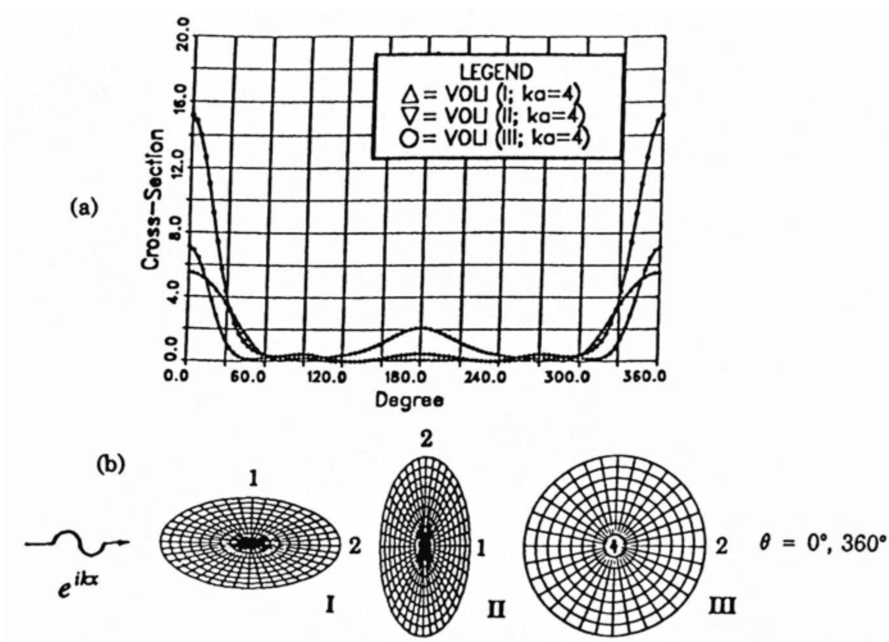
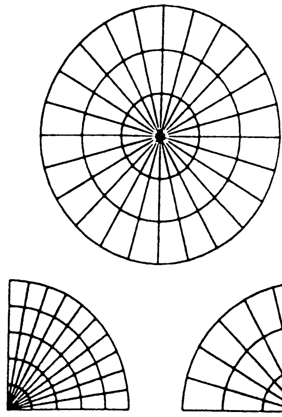
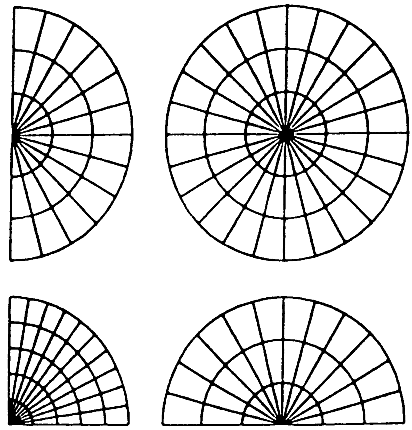


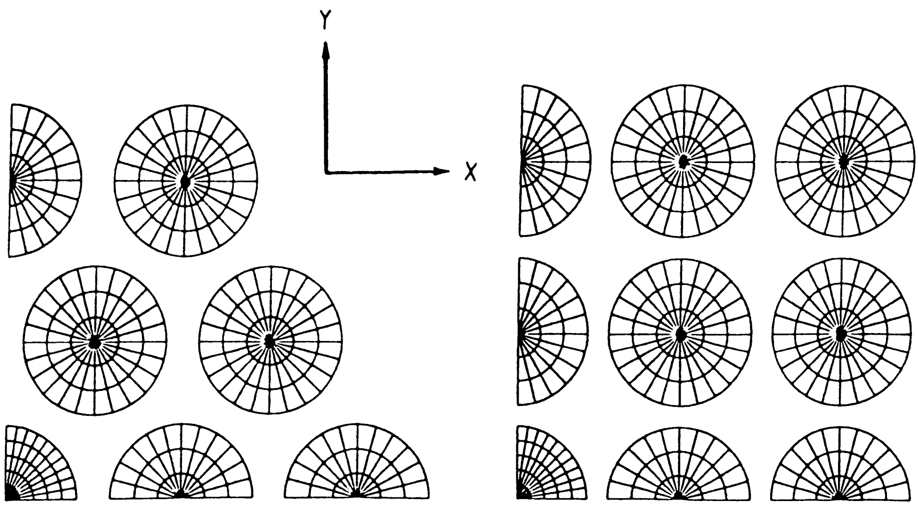
Fig. 5. Differential Cross–Section (a) for modeling in (b). Normalized wave number (ka) is 4. (k is the wavenumber in the host medium and a is the radius in the major axis.)



Average Strain	Re	Im
Analytical	0.09416	−0.0270
VIE	0.09474	−0.0272
Difference	0.61 %	0.74 %



Average Strain	Re	Im
Analytical	0.09035	−0.0329
VIE	0.09048	−0.0330
Difference	0.14 %	0.30 %



Average Strain	Re	Im
Analytical	0.08072	−0.0519
VIE	0.08089	−0.0523
Difference	0.21 %	0.77 %

Average Strain	Re	Im
Analytical	0.08047	−0.0645
VIE	0.08070	−0.0652
Difference	0.29 %	1.10 %

Fig. 6. Calculated average strains in the central fiber by analytical and volume integal equation (VIE) methods. The material is graphite/epoxy with volume concentration 0.6 and frequency is 10 MHz.

method, the symmetric and antisymmetric conditions are different from the standard finite element or the boundary integral equation method.

CONCLUSIONS

The volume integral equation method was developed as a new numerical solution scheme for SH wave scattering problems with arbitrary shapes and number of inclusions. Two methods, namely the volume integral equation method and the boundary integral equation method were compared for different types of problems.

For the single inclusion problem, both methods work very well. For multiple inclusions, the volume integral equation method gives very accurate results, is easier and more convenient, due to the fact that it is not necessary to apply continuity conditions at each interface. Furthermore, standard finite element pre−processors (e.g., Patran) can be used for geometric modeling, and the method is not sensitive to geometry, anisotropy, and inhomogeneity of inclusions. However, compared to the boundary integral equation method, the volume integral equation method is somewhat more C.P.U. intensive in terms of computer time.

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